Recall :

Def: Hilbert-Mumford weight

Let $x \in X(k)$, λ be a 1-PS (i.e. $\lambda: \mathbb{G}_m \to G$ nontrivial group homo.)

$G \curvearrowright X \subseteq \mathbb{P}^n$ linearly, so $G \curvearrowright \tilde{X} \subseteq \mathbb{A}^{n+1}$ also.

Let $\tilde{x} \in \tilde{X}$ be a nonzero lift of x.

Then there is a morphism

$$\lambda_{\tilde{x}}: \mathbb{G}_m \to \tilde{X}$$
$$t \mapsto \lambda(t) \cdot \tilde{x}.$$

By Prop 3.12, the action $\lambda(\mathbb{G}_m) \curvearrowright \mathbb{A}^{n+1}$ is linear $\Rightarrow$ diagonizable.

$\Rightarrow \exists$ basis $e_0, \dots, e_n$ of k^{n+1}, s.t.

$$\lambda(t) \cdot e_i = t^{r_i} e_i \quad , \ r_i \in \mathbb{Z}.$$

Suppose $\tilde{x} = \sum x_i e_i$, then

$$\lambda_{\tilde{x}}(t) = \lambda(t) \cdot \tilde{x} = \sum t^{r_i} x_i e_i .$$

The Hilbert-Mumford weight of x at λ

$$\mu(x, \lambda) := -\min(r_i : x_i \neq 0). \qquad \text{(Independent of choice of lift } \tilde{x}).$$

Today Exercise.

Exercise 6.5 (= Exercise sheet 10, Q1)

Exercise 1. Let G be a reductive group acting linearly on a projective scheme $X \subset \mathbb{P}^n$. For a 1-PS λ of G and $x \in X(k)$ and a non-zero lift $\tilde{x} \in \tilde{X}$, show that the Hilbert–Mumford weight has the following properties.

a) $\mu(x, \lambda)$ is the unique integer μ such that $\lim_{t \to 0} t^{\mu} \lambda(t) \cdot \tilde{x}$ exists and is non-zero.

b) $\mu(x, \lambda^n) = n\mu(x, \lambda)$ for positive integers n.

c) $\mu(g \cdot x, g\lambda g^{-1}) = \mu(x, \lambda)$ for all $g \in G(k)$.

d) $\mu(x, \lambda) = \mu(y, \lambda)$ where $y = \lim_{t \to 0} \lambda(t) \cdot x$.

(a) $\lim_{t \to 0} t^{\mu(x,\lambda)} \lambda(t) \cdot \tilde{x} = \lim_{t \to 0} t^{\mu(x,\lambda)} \sum t^{r_i} x_i e_i .$

$$= \sum \lim_{t \to 0} t^{r_i + \mu(x,\lambda)} x_i e_i$$

Using definition of $\mu(x,\lambda)$,

① For i s.t. $r_i = -\mu(x,\lambda)$, $t^{r_i+\mu} = t^0$.

$$\Rightarrow \lim_{t\to 0} t^{r_i+\mu(x,\lambda)} x_i e_i = x_i e_i$$

② For i s.t. $r_i \neq -\mu(x,\lambda)$, we must have $r_i + \mu > 0$ or $x_i = 0$.

(r_i is not minimum of λ-weights) (r_i is not a λ-weight)

$$\Rightarrow \lim_{t\to 0} t^{r_i+\mu(x,\lambda)} x_i e_i = 0.$$

$$\therefore \sum_i \lim_{t\to 0} t^{r_i+\mu(x,\lambda)} x_i e_i = \sum_{i:\, r_i = -\mu(x,\lambda)} x_i e_i \quad \text{exists.}$$

Since $\{e_i\}_i$ is a basis, limit value is nonzero.

Finally, to show uniqueness of μ,

Suppose $\mu' > \mu(x,\lambda)$, then Both ①,② would have $t^{\square}$ power > 0.

$$\Rightarrow \lim_{t\to 0} t^{\mu'} \lambda(t)\cdot \tilde{x} = 0.$$

Suppose $\mu' < \mu(x,\lambda)$, then ① would have $t^{\square}$ power < 0.

$$\Rightarrow \text{limit DNE.}$$

(b) $\lambda(t)\cdot e_i = t^{r_i} e_i \Rightarrow \lambda^n(t)\cdot e_i = t^{nr_i} e_i \quad \forall i$

$$\Rightarrow \mu(x,\lambda^n) = -\min(nr_i : x_i \neq 0).$$
$$= n\left(-\min(r_i : x_i \neq 0)\right)$$
$$= n\,\mu(x,\lambda).$$

(c) For $g \in G$, $\vdash: \mu(g.x,\ g\lambda g^{-1}) = \mu(x,\lambda)$

State: If λ is diagonal in basis e_i,
Then $g\lambda g^{-1}$ is diagonal in basis $g.e_i = e_i'$ } Think $g \in GL_n$.

$$(g\lambda g^{-1})_{\widetilde{g.x}}(t) = (g\lambda(t)g^{-1})\,\widetilde{g.x}$$
$$= (g\lambda(t)g^{-1})\cdot\left(\sum x_i\, g.e_i\right)$$
$$= \sum t^{r_i} x_i\, g\cdot e_i$$
$$= \sum t^{r_i} x_i\, e_i'$$

$\lambda(t)\cdot\tilde{x} = \sum t^{r_i} x_i e_i$ if e_i is the basis for λ to be diagonal.

Basis element for $g\lambda g^{-1}$ to be diagonal.

$\therefore \mu(g.x,\ g\lambda g^{-1}) = \mu(x,\lambda)$.

(d) Let $y = \lim_{t\to 0} \lambda(t)\cdot x$

$\tilde{x} = \sum x_i e_i$, $\tilde{y} = \sum y_i e_i$

$\lambda(t)\cdot \tilde{x} = \sum t^{r_i} x_i e_i$

By (a), $\lim_{t\to 0} t^{\mu} \lambda(t)\cdot \tilde{x}$ exist & nonzero

$\therefore$ μ is a lift for y., i.e. $\tilde{y} = \sum y_i e_i$, $y_i = \lim_{t\to 0} t^{\mu+r_i} x_i$,

$\Rightarrow y_i \neq 0$ iff $\mu = -r_i$ and $x_i \neq 0$

$\mu(y,\lambda) = -\min(r_i : y_i \neq 0)$.

$= -\min(r_i : x_i \neq 0)$

$= \mu(x,\lambda)$.

Exercise 2. Let SL_2 act on $V = k^2$ by left multiplication and consider the induced linear action of SL_2 on $\mathrm{Sym}^d(V)$. To simplify notation we identify $V \cong k[x,y]_1$ and $\mathrm{Sym}^d(V) \cong k[x,y]_d$. The goal of this exercise is to determine the (semi)stable k-points for the linear SL_2-action on $\mathbb{P}^{d+1} \cong \mathbb{P}(\mathrm{Sym}^d(V))$.

a) Show that any primitive 1-PS (that is, a 1-PS which is not a positive (power) multiple of any other 1-PS) of SL_2 is conjugate to the 1-PS $\lambda : \mathbb{G}_m \to \mathrm{SL}_2$

$$t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

b) Prove using Exercise 1, that a k-point $p \in \mathbb{P}^{d+1}$ is SL_2-semistable (resp. stable) if and only if $\mu(g\cdot p, \lambda) \geq 0$ (resp. > 0) for the above 1-PS λ and for all $g \in \mathrm{SL}_2(k)$.

c) Calculate the Hilbert–Mumford weight $\mu(p,\lambda)$ for any $p \in \mathbb{P}^{d+1}$ (for this, let $\tilde{p} = \sum_{i=0}^{d} a_i x^i y^{d-i}$ be a non-zero lift of p and note that the $\lambda(\mathbb{G}_m)$-action with respect to the basis $x^i y^{d-i}$ is diagonal).

d) Deduce that $\mu(p,\lambda) \geq 0$ if and only if $(1,0)$ occurs as a zero of $\sum_{i=0}^{d} a_i x^i y^{d-i}$ with multiplicity at most $d/2$. Determine an analogous statement for stability.

$G = SL_2 \curvearrowright \mathrm{Sym}^d(V) = k[x,y]_d$.

a) Let λ be a primitive 1-PS.

Then, (Prop 3.12) $\exists$ basis $\{e_1, e_2\}$ s.t. λ is diagonal,

i.e. $\lambda(t) \cdot e_i = t^{r_i} e_i$.

$\lambda(t) = \begin{bmatrix} t^{r_1} & \\ & t^{r_2} \end{bmatrix}$

Since $\lambda(t) \in SL_2$, $t^{r_1} \cdot t^{r_2} = 1$.

$\Rightarrow r_1 = -r_2$.

Since λ is primitive, $r_i = \pm 1$.

b) **Theorem 6.11.** *(Hilbert–Mumford Criterion) Let G be a reductive group acting linearly on a projective scheme $X \subset \mathbb{P}^n$. Then, for $x \in X(k)$, we have*

$x \in X^{ss} \iff \mu(x,\lambda) \geq 0$ for all 1-PSs λ of G,
$x \in X^{s} \iff \mu(x,\lambda) > 0$ for all 1-PSs λ of G.

WTS: $\mu(g.p, \lambda) \geq 0 \quad \forall \ g \in G$ for the fixed λ ——— ⊛

$\Leftrightarrow \mu(p, \lambda') \geq 0 \quad \forall \ \lambda'$ 1-PS.

By Q1(c), $\mu(g.p, \lambda) = \mu(p, g^{-1}\lambda g)$.

As $g \in G$ varies, $g^{-1}\lambda g$ runs through all primitive 1-PS by part (a).

$\therefore \mu(p, \lambda') \geq 0 \quad \forall \ \lambda'$ primitive 1-PS.

Every 1-PS is a positive multiple (power) of some primitive 1-PS λ'.

Say $\alpha = (\lambda')^n$.

By Q1(b), $\mu(p, \alpha) = n\mu(p, \lambda')$

≥ 0 iff $\mu(p, \lambda') \geq 0$

$\therefore \mu(p, \alpha) \geq 0 \quad \forall \ \alpha$ 1-PS.

Backward implication:

Use fact that Every 1-PS has a primitive representative.

By Hilbert-Mumford, ⊛ iff p is semistable.

for stability argument: Replace all ≥ 0 to > 0

(c) Calculate $\mu(p, \lambda)$.

$\lambda(t) = \begin{bmatrix} t & \\ & t^{-1} \end{bmatrix}$

Let $\tilde{p} = \sum_{i=0}^{d} a_i x^i y^{d-i}$ be a nonzero lift of p.

$\lambda(t) \cdot \tilde{p} = \sum_{i=0}^{d} a_i t^{-i} \cdot t^{d-i} x^i y^{d-i}$ since $\lambda(t)$ acts on $\tilde{p}$ by mult of $\lambda(t^{-1})$

$= \sum_{i=0}^{d} t^{d-2i} a_i x^i y^{d-i}$

$\mu(p,\lambda) = -\min(d-2i \;,\; a_i \neq 0)$

$= -d + 2\max(i : a_i \neq 0)$

d) Let m be the multiplicity of $(1,0)$ as a zero of $\sum_{i=0}^{d} a_i x^i y^{d-i}$. ($m \geq 0$, can be 0.)

$\Leftrightarrow$ $y^m \mid \sum_{i=0}^{d} a_i x^i y^{d-i}$ and $y^{m+1} \nmid \sum_{i=0}^{d} a_i x^i y^{d-i}$.

$\Updownarrow$ $\Updownarrow$

$d-i \geq m$ for all i s.t. $a_i \neq 0$ $\quad$ $a_i \neq 0$ for $d-i = m$.

$\Updownarrow$ $\Updownarrow$

If $d-i < m$, then $a_i = 0$. $\quad$ $a_{d-m} \neq 0$.

$\Updownarrow$

$a_i = 0$ for $i > d-m$.

a_{d-m} is the last nonzero coeff.

$\mu(p,\lambda) = -\min(d-2i \;,\; a_i \neq 0)$

$\leq d-m$, equals when min attained.

$\Rightarrow \mu(p,\lambda) = -(d - 2(d-m))$

$= d - 2m$

$\therefore$ $\mu(p,\lambda) \geq 0 \Leftrightarrow d-2m \geq 0$

$\Leftrightarrow m \leq \frac{d}{2}$ //

Still holds

For stability, $d-2m > 0$,

$m < \frac{d}{2}$.

e) Determine the (semi)stability of p in terms of the multiplicities of the zeros of $\sum_{i=0}^{d} a_i x^i y^{d-i}$ in $\mathbb{P}^1$.

f) For $d = 3, 4$, determine the GIT quotient $\mathbb{P}^{d+1}//\mathrm{SL}_2$ and the geometric quotient of the stable locus (recall the closed points of the GIT quotient $\mathbb{P}^{d+1}//\mathrm{SL}_2$ are in bijection with the polystable orbits).

g) (Optional/Harder.) Determine a moduli problem for which the GIT quotient $\mathbb{P}^{d+1}//\mathrm{SL}_2$ is a coarse moduli space.

(e) By part (b), Want $\mu(g.p, \lambda) \geq 0 \quad \forall \ g \in G$ for semistability.

$\Updownarrow$ part (d)

$(1,0)$ is zero of $g.\sum_{i=0}^{d} a_i x^i y^{d-i}$ w/ mult. $\leq \frac{d}{2} \quad \forall g \in G$

↑ action by inverse

$\Updownarrow$

no inverse

$g.(1,0)$ is zero of $\sum_{i=0}^{d} a_i x^i y^{d-i}$ w/ mult. $\leq \frac{d}{2} \quad \forall g \in G$

Any point, as g varies.

$\Updownarrow$

All roots of $\sum_{i=0}^{d} a_i x^i y^{d-i}$ has mult. $\leq \frac{d}{2}$.

$\therefore$ p is a (semi) stable point if

all zeros of $\sum_{i=0}^{d} a_i x^i y^{d-i}$ have multiplicity $< \frac{d}{2}$.

$(\leq \frac{d}{2})$.