Quick summary for §5.3 - §5.5:

Recall in §5.1 - §5.2,

For X projective, reductive $G \curvearrowright X$,

Suppose we have $X \hookrightarrow \mathbb{P}^n$ & $G \curvearrowright GL_{n+1} \rightsquigarrow G \curvearrowright \mathbb{P}^n$

s.t. inclusion is compatible w/ G-action.

↑ We call this linear action of G on $X \subseteq \mathbb{P}^n$

We can define projective GIT.

§5.4: Q: How to get $X \hookrightarrow \mathbb{P}^n$ that is compatible w/ G-action?

A: Ample line bundle!

Let L be an ample line bundle on X.

(Hartshorne II.7, P.153, see also II.5 P.120 for very ample)

§5.3 $\rightsquigarrow$ Have an ample linearization (Def 5.15).

This gives a G-equivariant closed embedding

$$X \hookrightarrow \mathbb{P}\, H^0(X,L)^*$$

Note: $H^0(X,L)$ is a k vector space w/ a G action.

§5.4 $\rightsquigarrow$ Defined (semi)stability that agree w/ the defin. in §5.1.

and constructed GIT quotient.

§5.5: Let X quasi projective (more general than affine/proj. GIT)

L be a linearization. (not necessarily ample)

Can similarly do GIT.

§6 Setting: Let X projective, G reductive, $G \curvearrowright X$.

Assume we have a linear action of G on $X \subseteq \mathbb{P}^n$

Want: Let $x \in X(k)$ (closed point), determine if x is (semi)stable.

§6.1 Topological criterion

Let $\tilde{X} \subseteq \mathbb{A}^{n+1}$ be the affine cones over $X \subseteq \mathbb{P}^n$.

$G \curvearrowright \mathbb{P}^n \rightsquigarrow G \curvearrowright \mathbb{A}^{n+1}$.

Let $R(X) = O(\tilde{X})$ be the homog. coord. ring of X.

Prop 6.1 Let $x \in X(k)$ and choose a lift $\tilde{x} \in \tilde{X}(k)$ of x. Then:

(i.e. if $x = [x_0 : \cdots : x_n] \in \mathbb{P}^n$, $\tilde{x} := (x_0, \ldots, x_n) \in \mathbb{A}^{n+1}$)

i) x is semistable $\Leftrightarrow 0 \notin \overline{G \cdot \tilde{x}}$

ii) x is stable $\Leftrightarrow \dim G_{\tilde{x}} = 0$ and $G \cdot \tilde{x}$ is closed in $\tilde{X}$.

Recall: Def
- x is semistable if $x \notin V(R(X)^G_+)$
- x is stable if $\begin{cases} \dim G_x = 0 \\ \exists\, f \in R(X)^G_+ \text{ s.t. } x \in X_f \text{ and } G \curvearrowright X_f \text{ is closed.} \end{cases}$

Proof (i)($\Rightarrow$): Suppose x is ss.

Then $\exists\, f \in R(X)^G_+$ homog. s.t. $f(\tilde{x}) \neq 0$.

View f as function on $\tilde{X}$ gives $f(\tilde{x}) \neq 0$.

Note: f is G-inv and constant on orbit closures

$\therefore f(\overline{G \cdot \tilde{x}}) \neq 0$

But $f(0) = 0$

$\therefore 0 \notin \overline{G \cdot \tilde{x}}$

Fact: f const. on orbit closures

Pf: G-inv $\Rightarrow$ const. on orbits.

Topology fact: $f(\overline{X}) \subseteq \overline{f(X)}$

$\therefore f(\overline{X})$ is a point.

$X \subseteq f^{-1}(\overline{f(X)}) \Rightarrow \overline{X} \subseteq f^{-1}(\overline{f(X)})$

$\Rightarrow f(\overline{X}) \subseteq \overline{f(X)}$

(i) ($\Leftarrow$): Suppose $0 \notin \overline{G\cdot\tilde{x}}$.

G geom. reductive $\overset{4.29}{\Rightarrow} \exists\, f \in R(X)^G$ s.t. $f(0)=0$, $f(\overline{G\cdot\tilde{x}})=1$.

(Also want f to be homog.)

Let $f = f_0 + \cdots + f_r$, $\begin{cases} G \curvearrowright \mathbb{P}^n \text{ linear} \\ G\cdot f = f \end{cases} \Rightarrow G\cdot f_i = f_i \ \forall i$

In particular $f_r(x) \neq 0 \Rightarrow x \notin V(R(X)^G_+)$ semistable.

(ii) ($\Rightarrow$): Suppose X stable.

$G_{\tilde{x}} \subseteq G_x$. $\therefore \dim G_x = 0 \Rightarrow \dim G_{\tilde{x}} = 0$.

($\because g\cdot\tilde{x}=\tilde{x} \Rightarrow g\cdot x = x$)

Also, $\exists\, f \in R(X)^G_+$ homog. s.t. $\begin{cases} x \in X_f \Rightarrow f(x)\neq 0 \\ G\cdot x \text{ is closed in } X_f. \end{cases}$

Again view f as a function on $\tilde{X}$. $f(\tilde{x}) = f(x) \neq 0$

Let $Z := \{ z \in \tilde{X} : f(z) = f(\tilde{x}) \} \subseteq \tilde{X}$ (closed, G-inv)

Suffices to show that $G\cdot\tilde{x} \subseteq Z$ is closed:

Pf: We have:

$$\begin{array}{ccc} Z & \overset{\text{codim } 1}{\hookrightarrow} & \tilde{X} - \{0\} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ X_f & \hookrightarrow & X \end{array}$$

(Z: $\dim X$; $\tilde{X}-\{0\}$: $\dim X + 1$; X_f: $\dim X$; X: $\dim X$)

$\therefore \pi : Z \to X_f$ finite.

$G\cdot x|_{X_f}$ closed in X_f

$\Rightarrow \pi^{-1}(G\cdot x|_{X_f})$ closed in Z

π is finite $\Rightarrow$ $\pi^{-1}(G\cdot x)$ is a finite number of orbits. (lifts of $G\cdot x$)

$\Rightarrow$ All orbits in $\pi^{-1}(G\cdot x)$ are of dimension equal to $\dim G$.

$\Rightarrow$ All such orbits are closed (in Z)

Otherwise if $G\cdot z$ not closed,

have orbit $\subseteq \overline{G\cdot z} - G\cdot z$ of dimension $< \dim G$.

In particular $G\cdot\tilde{x} \subseteq Z$ closed

(ii)($\Leftarrow$): Suppose $\dim G_{\tilde{x}} = 0$ and $G\cdot\tilde{x}$ closed in $\tilde{X}$.

Then $0 \notin G\cdot\tilde{x} = \overline{G\cdot\tilde{x}} \overset{(i)}{\Rightarrow} x$ is semi stable.

Take any $f \in R(X)^G_+$ homog. s.t. $f(x) \neq 0$.

Let $\pi: Z := \{z \in \tilde{X} : f(z) = f(\tilde{x})\} \to X_f$

$$\begin{cases} \pi \text{ is finite} \Rightarrow \begin{cases} \dim G_x = 0 \\ G\cdot x \text{ closed in } X_f \end{cases} \\ \pi(G\cdot\tilde{x}) = G\cdot x \end{cases}$$

Since $G\cdot x$ closed in X_f for all $f \in R(X)^G_+$ s.t. $f(x) \neq 0$,

and $X^{ss} = \bigcup_f X_f$

$\therefore$ $G\cdot x$ closed in X^{ss} by Lemma 5.9

$\therefore$ x is stable.

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§ 6.2 Hilbert-Mumford criterion

Question: $\overline{G\cdot x} = ?$

Rmk: $0 \notin \overline{G\cdot\tilde{x}} \Rightarrow x$ is ss

$\dim G_x = 0$ & $\overline{G\cdot\tilde{x}} = G\cdot\tilde{x} \Rightarrow x$ stable

Def • A 1-parameter subgroup (1-PS) of G is a non-triv. group hom.

$$\lambda: \mathbb{G}_m \longrightarrow G$$

• Fix $x \in X(k)$ and 1-PS $\lambda: \mathbb{G}_m \to G$.

Then let $\lambda_x: \mathbb{G}_m \longrightarrow X$ by

$$t \longmapsto \lambda(t) \cdot x$$

Note: • Have embedding $\mathbb{G}_m = \mathbb{A}^1 - \{0\} \to \mathbb{P}^1$ by

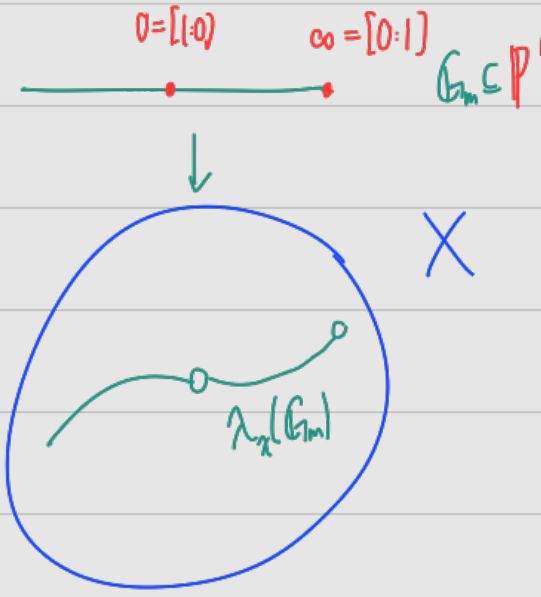
$$t \longmapsto [1:t]$$

• X projective $\Rightarrow$ X proper over $\operatorname{Spec} k$ (Def: Hartshorne II.4)

(valuative criterion for properness) $\Rightarrow \exists!\ \hat{\lambda}_x: \mathbb{P}^1 \to X$ s.t.

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\lambda_x} & X \\ \downarrow & \nearrow_{\hat{\lambda}_x} & \downarrow \\ \mathbb{P}^1 & \longrightarrow & \operatorname{Spec} k \end{array}$$

Picture:



There exists a unique way to "fill in the holes".

Let $\lim_{t\to 0} \lambda_x(t) = \hat{\lambda}_x([1:0])$. $\lim_{t\to\infty} \lambda_x(t) = \hat{\lambda}_x([0:1])$.

$\lim_{t\to 0} \lambda(t)\cdot x$

Rmk: ① $\lambda_x(G_m) \subseteq S \Rightarrow \hat{\lambda}_x(\mathbb{P}^1) \subseteq \overline{S}$.

② The limit of λ at ∞ is the limit of λ^{-1} (another 1-PS) at 0.

$\therefore$ Can just look at $\lim_{t\to 0} \lambda_x(t)$.

③ Let $y := \lim_{t\to 0} \lambda_x(t)$.

Then $y \in \overline{\lambda(G_m)\cdot x} - \lambda(G_m)\cdot x$

$\begin{cases} \lambda(G_m)\cdot x \text{ is 1-dim} \\ \overline{\lambda(G_m)\cdot x} - \lambda(G_m)\cdot x \text{ is a union of lower dim orbits} \end{cases} \Rightarrow \lambda(G_m)$ fixes y.

Definition 6.3 (Skipped)

Look at affine cone $\tilde{X}$:

Pick non-zero lift $\tilde{x} \in \tilde{X}$ of $x \in X$.

Have: $\lambda_{\tilde{x}} : G_m \to \tilde{X}$.

Note:
- $\tilde{X}$ not proper, may no longer extend.
- If can extend at 0 or ∞, denote the limits similarly
- Any points on the boundary $\overline{\lambda_{\tilde{x}}(G_m)} - \lambda_{\tilde{x}}(G_m)$ must be either of the limits

$G \curvearrowright \mathbb{A}^{n+1}$ linear $\Rightarrow$ $\lambda(G_m) \curvearrowright \mathbb{A}^{n+1}$ linear $\overset{(\text{Prop } 3.12)}{\Longrightarrow}$ Weight-space decomp:

Can pick basis $e_0, \dots, e_n \in \mathbb{A}^{n+1}$ s.t.

$\lambda(t) \cdot e_i = t^{r_i} e_i$ for $r_i \in \mathbb{Z}$.

* Call r_i the λ-weights of the action $\lambda(G_m)$ on $\mathbb{A}^{n+1}$.

Write $\tilde{x} = \sum_{i=0}^{n} x_i e_i$.

then $\lambda(t) \cdot \tilde{x} = \sum_{i=0}^{n} t^{r_i} x_i e_i$.

* Let $\lambda\text{-wt}(x) := \{r_i : x_i \neq 0\}$ be the λ-weights of x

↑ only depends on x, doesn't depend on choice of $\tilde{x}$.

Def 6.4 We define the Hilbert-Mumford weight of λ at x to be

$$\mu(x, \lambda) := -\min\{r_i : x_i \neq 0\}.$$

e.g. (Exercise sheet 9): $G = (G_m)^2 \curvearrowright \mathbb{P}^3$ by the rep:

$$G \longrightarrow GL_4$$
$$(s,t) \longmapsto \operatorname{diag}(st, s^{-1}t, s^{-1}t^{-1}, st^{-1}) \quad \text{(already diagonalized)}$$

Random points on $\mathbb{P}^3$: $x = [1:0:2:4]$, $[1:2:3:4]$, $[1:1:0:0]$

Pick $\tilde{x} = (1,0,2,4)$

	$\lambda(s) = (s,1)$	$\lambda(t) = (1,t)$
$x = [1:0:2:4]$	$\lambda\text{-wt}(x) = \{1, -1, 1\}$, $\mu(x,\lambda) = 1$	$\lambda\text{-wt}(x) = \{1, -1, -1\}$, $\mu(x,\lambda) = 1$
$[1:2:3:4]$	$\lambda\text{-wt}(x) = \{1, -1, -1, 1\}$, $\mu(x,\lambda) = 1$	$\lambda\text{-wt}(x) = \{1, 1, -1, -1\}$, $\mu(x,\lambda) = 1$
$[1:1:0:0]$	$\lambda\text{-wt}(x) = \{1, -1\}$, $\mu(x,\lambda) = 1$	$\lambda\text{-wt}(x) = \{1, 1\}$, $\mu(x,\lambda) = -1$

Exercise/Facts 6.5

1) $\mu(x,\lambda)$ is the unique integer s.t. $\lim_{t\to 0} t^{\mu}\lambda(t)\cdot\tilde{x}$ exists and $\neq 0$.
2) $\mu(x,\lambda^n) = n\,\mu(x,\lambda)$ for $n>0$.
3) $\mu(g\cdot x, g\lambda g^{-1}) = \mu(x,\lambda)$ $\forall g\in G$
4) $\mu(x,\lambda) = \mu(y,\lambda)$ where $y = \lim_{t\to 0}\lambda(t)\cdot x$

Lemma 6.7

Suppose the $\lambda(G_m)$-action is diagonalized.

(i) $\mu(x,\lambda)<0 \Leftrightarrow \tilde{x} = \sum_{r_i>0} x_i e_i \Leftrightarrow \lim_{t\to 0}\lambda(t)\cdot\tilde{x} = 0$

(ii) $\mu(x,\lambda)=0 \Leftrightarrow \tilde{x} = \sum_{r_i\geq 0} x_i e_i$ & $\exists\, r_i = 0$ s.t. $x_i\neq 0$
$\Leftrightarrow \lim_{t\to 0}\lambda(t)\cdot\tilde{x}$ exists and $\neq 0$

say $r_0 = 0$, $x_0\neq 0$
then $t\cdot\tilde{x} = x_0 e_0 + \sum_{i=1}^{n} t^{r_i} x_i e_i \neq 0$

(iii) $\mu(x,\lambda)>0 \Leftrightarrow \tilde{x} = \sum_{r_i\geq 0} x_i e_i$ & $\exists\, r_i<0$ s.t. $x_i\neq 0$
$\Leftrightarrow \lim_{t\to 0}\lambda(t)\cdot\tilde{x}$ DNE

say $r_0 = -1$, $x_0\neq 0$
then $t\cdot\tilde{x} = t^{-1}x_0 e_0 + \sum_{i=1}^{n} t^{r_i} x_i e_i \neq 0$

Rmk 6.8

Have analog for $\lim_{t\to\infty}$

Combining §6.1, have (semi)stability for a 1-PS action.

Lemma 6.9

(i) x is ss for $\lambda(G_m) \Leftrightarrow \mu(x,\lambda)\geq 0$ & $\mu(x,\lambda^{-1})\geq 0$.

(ii) x is stable for $\lambda(G_m) \Leftrightarrow \mu(x,\lambda)>0$ & $\mu(x,\lambda^{-1})>0$.

Proof: (i) x is ss $\overset{\text{Prop 6.1}}{\Leftrightarrow} 0\notin \overline{G\cdot\tilde{x}} \Leftrightarrow \lim_{t\to 0}\lambda(t)\cdot\tilde{x}\neq 0$ or DNE
$\lim_{t\to\infty}\lambda(t)\cdot\tilde{x}\neq 0$
$\Leftrightarrow \begin{cases}\mu(x,\lambda)\geq 0\\ \mu(x,\lambda^{-1})\geq 0\end{cases}$

(ii) x stable $\Leftrightarrow$ $\begin{cases} \dim \lambda(\mathbb{G}_m)_{\tilde{x}} = 0 \\ \lambda(\mathbb{G}_m)\cdot\tilde{x} \text{ closed} \end{cases}$ $\Leftrightarrow$ limits DNE $\Leftrightarrow$ $\begin{cases} \mu(x,\lambda)>0 \\ \mu(x,\lambda^{-1})>0 \end{cases}$

($\lambda(\mathbb{G}_m)$ cannot fix x; $\Leftarrow$)

Rmk: Lemma 6.9 gives

x ss $\Rightarrow \mu(x,\lambda) \geq 0$ $\forall$ 1-PS λ of G

x stable $\Rightarrow \mu(x,\lambda) > 0$ $\forall$ 1-PS λ of G

Thm 6.11 (Hilbert-Mumford criterion): The converses of above are true.

Rmk 6.12 λ is primitive if $\nexists$ 1-PS τ, $n\in\mathbb{Z}_{>0}$ s.t. $\lambda = \tau^n$.

Suffices to check H-M on all primitive 1-PS's of G.

To show 6.11, suffices to show there are "enough 1-PS's":

Thm 6.13 (Fund. thm. of GIT) (6.13 implies H-M criterion)

Let G reductive w/ $G \curvearrowright \mathbb{A}^{n+1}$.

Let $\tilde{x}\in\mathbb{A}^{n+1}$ closed pt, $y \in \overline{G\cdot\tilde{x}}$,

then $\exists$ 1-PS λ of G s.t. $\lim_{t\to 0}\lambda(t)\cdot\tilde{x} = y$.