Moduli Reading Group Week 11: 22nd Oct '26.

Construction of the Projective GIT Quotient.

– Pijush

- **Motivation:**

– Recall Affine GIT Quotient:

$G \curvearrowright$ Reductive grp.
Acting on an affine scheme X.
$\mathcal{O}(X)^G$ - f.g. (Nagata).

Affine GIT Quotient is the morphism $\varphi : X \to X/\!\!/_G := \operatorname{Spec} \mathcal{O}(X)^G$, coming from the ring morphisms $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$.

– Today we will do this for projective schemes $X \subseteq \mathbb{P}^n$.

– Establish analogous properties.

Setup: $G \curvearrowright$ affine alg group. $X \curvearrowright$ projective scheme.

Defn 5.1: A linear G-equivariant projective embedding of X is a:

(i) Group homomorphism $G \to GL_{n+1}$
(ii) G-equivariant proj embedding $X \hookrightarrow \mathbb{P}^n$

• Suppose we have a linear G-action (→ reductive.) on $X \hookrightarrow \mathbb{P}^n$, with

- $I(X) \rightsquigarrow$ homog ideal of X

  i.e. the affine cone over X is $\tilde{X} := \operatorname{Spec}\left(R(X) := k[x_0, \dots, x_n]/I\right)$

- $R(X) := \bigoplus_{r \geq 0} R(X)_r \rightsquigarrow$ Graded k-alg.

- G-action on $X \hookrightarrow \mathbb{P}^n$ is linear $\Rightarrow R(X)^G = \bigoplus_{r \geq 0} R(X)_r^G$

- Nagata $\Rightarrow$ $R(X)^G$ is f.g.

- Any inclusion of graded f.g k-alg gives a rational map from corresponding proj schemes. Thus we get

$$X := \operatorname{Proj}(R(X)) \dashrightarrow \operatorname{Proj}(R(X)^G).$$

- $R(X)^G_+ := \bigoplus_{r > 0} R(X)_r^G$ is a homog ideal

  It determines the closed subscheme where the rational map is not defined.

Q: What is the domain of $X \dashrightarrow \operatorname{Proj} R(X)^G$?

As we just saw

Domain = $X \setminus$ closed subscheme determined by $R(X)^G_+$

= $X \setminus N$ → called the null cone of X

Define $X^{ss} := X \setminus N$

This is called the semistable set, and it is the domain of the rational map $X \dashrightarrow \operatorname{Proj}(R(X)^G)$.

- The morphism $X^{ss} \longrightarrow \operatorname{Proj} R(X)^G := X/\!/G$ is called GIT quotient of the linear G-action on $X \hookrightarrow \mathbb{P}^n$.

Recall Affine GIT Quotient: The map $\varphi: X \longrightarrow X/\!/G$ was defined on all of X and it turned out to be a good quotient.

In projective GIT Quot, we have:

Theorem 5-3: Suppose G-reductive with a linear G-action on $X \hookrightarrow \mathbb{P}^n$. Then the GIT quotient $\varphi: X^{ss} \longrightarrow X/\!/G$ is a good quotient of the G-action on X^{ss}.
Also, $X/\!/G$ is a projective scheme.

Proof idea: First part: gluing.

- We know being a good quotient is local on target.
- Denote $Y := \operatorname{Proj} R(X)^G$. Then $\{D_+(f) : f \in R(X)^G_+\}$ cover Y.
- For each $f \in R(X)^G_+$, consider Y_f and $X_f := \varphi^{-1}(Y_f)$.
- Then $\varphi_f: X_f \longrightarrow Y_f$ is an affine GIT quotient.

  Indeed, $\mathcal{O}(Y_f) \cong \mathcal{O}(X_f)^G$

  So the inclusion

$$\mathcal{O}(X_f)^G \hookrightarrow \mathcal{O}(X_f) \quad \text{induces}$$

$$X_f = \operatorname{Spec}(\mathcal{O}(X_f)) \xrightarrow{\varphi_f} \operatorname{Spec}(\mathcal{O}(X_f)^G) \cong \operatorname{Spec}(\mathcal{O}_{Y_f}) = Y_f.$$

- Thus, $\varphi_f: X_f \longrightarrow Y_f$ is a good quotient.

2nd Part: Thm and exercise in Hartshorne. ▨

- By virtue of $\varphi: X^{ss} \longrightarrow X/\!/G$ being a good quotient, the following hold:
  (i) $\varphi(x_1) = \varphi(x_2) \iff \overline{G\cdot x_1} \cap \overline{G\cdot x_2} \cap X^{ss} \neq \emptyset$
  (ii) $\forall\, y \in X/\!/G$. $\varphi^{-1}(y)$ contains a unique closed orbit.
  $\therefore$ Action closed $\Rightarrow$ geometric quotient.

- It is the presence of non-closed orbits in X^{ss} that prevents $\varphi: X^{ss} \longrightarrow X/\!/G$ from being a geometric quotient.

Analogously to affine GIT-Quot, we ask:

Q: Is there an open set in X^{ss} where φ is geometric?

Defn 5.4: A point $x \in X$ is
(i) Stable if $G_x = 0$ and $\exists$ G-inv homog poly $f \in R(X)^G_+$ such that $x \in X_f$ and the action of G on X_f is closed.
(ii) Unstable if not semi-stable, i.e., not in X^{ss}.
(iii) $X^{ss} := X \setminus N$ = set of semi-stable pts } Open
X^s: set of stable points
$X^{us} := X \setminus X^{ss} = N$ set of unstable points — closed.

- Theorem 5.6: For the GIT quotient $\varphi: X^{ss} \longrightarrow Y := X/\!/G$, there is an open subscheme $Y^s \subset Y$ such that
(i) $\varphi^{-1}(Y^s) = X^s$
(ii) the restriction $\varphi|_{X^s}: X^s \longrightarrow Y^s$ is geometric quotient.

Proof idea:

- Construction of Y^s.

- $Y_c := \bigcup Y_f$ where the union is over all $f \in R(X)^G_+$ s.t. G-action on X_f is closed.

- If $X_c := \bigcup X_f$, then $X_c = \varphi^{-1}(Y_c)$.

- Construct $\varphi: X_c \to Y_c$ by gluing $\varphi_f: X_f \to Y_f$.

- Each φ_f geom quot $\Rightarrow$ $\varphi: X_c \to Y_c$ is geom quot.

- Define $Y^s := \varphi(X^s)$. By construction $X^s \underset{\text{open}}{\subset} X_c$ and thus, $Y^s \subset Y_c$ giving $\varphi: X^s \to Y^s$ geometric.

- Y^s is open: • Indeed, $Y^s \subset Y_c \underset{\text{open}}{\subset} Y$. So suffices to prove Y^s is open in Y_c.
  • $Y_c \setminus Y^s = \varphi(X_c \setminus X^s)$.
  • $X_c \setminus X^s$ closed and by property of good quotients, so is $\varphi(X_c \setminus X^s)$.

Example 5.8: Linear action of $G = \mathbb{G}_m := \operatorname{Spec} k[t, t^{-1}]$ on $X = \mathbb{P}^n$ by $t \cdot [x_0 : \dots : x_n] = [t^{-1}x_0 : tx_1 : \dots : tx_n]$.

- $R(X) = \dfrac{k[x_0, \dots, x_n]}{I(X)} = k[x_0, \dots, x_n] \rightsquigarrow$ graded by homog degree

$I(X)$ → homog ideal corresponding to $\mathbb{P}^n = 0$

$I(X) = \langle f \text{ homog} \mid f([x_0 : \dots : x_n]) = 0 \ \forall \ x \in \mathbb{P}^n \rangle$

- Functions $x_0x_0, \dots, x_0x_n$ are G-inv:

Indeed, take $f([x_0 : \dots : x_n]) = x_0x_i$

Then $t \cdot f([x_0 : \dots : x_n]) = f(t^{-1} \cdot [x_0 : \dots : x_n])$

$= f(t x_0 : \dots : t^{-1}x_n)$

$= t x_0 t^{-1} x_i$

$= x_0 x_i$

$= f.$

- Claim: $R(X)^G := \text{ring of invariants} = k[x_0x_1, \dots, x_0x_n]$.

Suppose $f \in R(X)^G$, write

$$f = \sum_{\underline{m}} a_{\underline{m}} x_0^{m_0} \cdots x_n^{m_n}$$

Then $t \cdot f = f \Rightarrow$

$$\sum_{\underline{m}} a_{\underline{m}} t^{m_0 - m_1 - \cdots - m_n} x_0^{m_0} \cdots x_n^{m_n} = \sum_{\underline{m}} a_{\underline{m}} x_0^{m_0} \cdots x_n^{m_n}$$

$\Rightarrow \forall \ \underline{m} = (m_0, \dots, m_n)$: either $a_{\underline{m}} = 0$

or $t^{m_0 - m_1 - \cdots - m_n} = 1$

i.e. $m_0 = m_1 + \cdots + m_n$

$$\Rightarrow \quad x_0^{m_0} x_1^{m_1} \cdots x_n^{m_n} = (x_0x_1)^{m_1} \cdots (x_0x_n)^{m_n}$$

Thus, $f \in k[x_0x_1, \ldots, x_0x_n]$ and hence

$$R(X)^G = k[x_0x_1, \ldots, x_0x_n] \cong k[y_0, \ldots, y_{n-1}]$$

- This gives $X/\!/_G := \operatorname{Proj} R(X)^G = \operatorname{Proj} k[y_0, \ldots, y_{n-1}] = \mathbb{P}^{n-1}$.

- What is the GIT quotient ?

$$\varphi : X = \mathbb{P}^n \dashrightarrow X/\!/_G = \mathbb{P}^{n-1}$$

$$[x_0 : \cdots : x_n] \longmapsto [x_0x_1 : \cdots : x_0x_n]$$

- Clearly, null-cone, $N = \{ [x_0 : \cdots : x_n] \in \mathbb{P}^n \mid x_0 = 0 \text{ or } x_1 = \cdots = x_n = 0 \}$

$$= V(x_0x_1, \ldots, x_0x_n).$$

$$X^{ss} = X \setminus N$$

$$= \mathbb{P}^n \setminus \left(\mathbb{P}^n_{x_0x_1} \cup \cdots \cup \mathbb{P}^n_{x_0x_n} \right)$$

$$= \text{First move to the chart } x_0 \neq 0 \} \cong \mathbb{A}^n$$

and then

$$(x_1, \ldots, x_n) \neq 0 \} \ 0 \in \mathbb{A}^n$$

$$\cong \mathbb{A}^n \setminus 0.$$

- Good quotient map $\varphi : X^{ss} \cong \mathbb{A}^n - 0 \longrightarrow \mathbb{P}^{n-1}$

$$X^{ss} = \mathbb{P}^n \cap \left\{ [x_0 : \cdots : x_n] \;\middle|\; \begin{matrix} x_0 \neq 0 \\ (x_1, \ldots, x_n) \neq (0, \ldots, 0) \end{matrix} \right\}$$

Thus $X^{ss} \cong \mathbb{A}^n - \{0\}$ is given by

$$[x_0 : \cdots : x_n] \longmapsto \left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right)$$

So $\varphi : X^{ss} \longrightarrow \mathbb{P}^{n-1}$ is $[x_0 : \cdots : x_n] \longmapsto \left[\frac{x_1}{x_0} : \cdots : \frac{x_n}{x_0}\right]$.

- Pre-image of $\left[\frac{x_1}{x_0} : \cdots : \frac{x_n}{x_0}\right]$ = Pre-image of $\left[\frac{t^2 x_1}{x_0} : \cdots : \frac{t^2 x_n}{x_0}\right]_{t \neq 0}$

$$= \text{Pre-image of } \left[\frac{t x_1}{t^{-1} x_0} : \cdots : \frac{t x_n}{t^{-1} x_0}\right]$$

$$= \left[t^{-1} x_0 : t x_1 : \cdots : t x_n\right]$$

$$= \left\{ t \cdot [x_0 : x_1 : \cdots : x_n] \;\middle|\; t \neq 0 \right\}$$

Single orbit. Therefore Geom quot.

- Orbits in $X^{ss} \cong \mathbb{A}^n - \{0\}$ are $\left\{ \left(\frac{t^2}{x_0} x_1, \cdots, \frac{t^2}{x_0}, x_n\right) \;\middle|\; t \neq 0 \right\}$

$$= \left\{ (\lambda x_1, \cdots, \lambda x_n) \;\middle|\; \lambda \neq 0 \right\}$$

$$= (\text{line through } \bar{x}) \cap (\mathbb{A}^n - \{0\})$$

$$= \text{closed in } X^{ss} = \mathbb{A}^n - \{0\}$$

$$\text{Stabilizer} = \left\{ t \in k^\times \;\middle|\; \left(\frac{t^2 x_1}{x_0}, \cdots, \frac{t^2 x_n}{x_0}\right) = \left(\frac{x_1}{x_0}, \cdots \frac{x_n}{x_0}\right) \right\}$$

$$= \{ t^2 = 1 \} \Rightarrow \pm 1 \longrightarrow 0\text{-dim.}$$

$\therefore \quad X^{ss} = X^s$.

In the example above: we determined stable and semi-stable points by using information of $R(X)^G$.

Because $X^{ss} = X \setminus N$

↑ determined by $R(X)^G_+$

However determining $R(X)^G_+$ is hard. So this motivates other criteria of determining (semi) stability.

Lemma 5.9: $x \in X(k)$ stable $\Leftrightarrow$ (i) x is semistable
(ii) $G \cdot x$ is closed in X^{ss}
(iii) $\dim G_x = 0$

Defn 5.12: $x \in X(k)$ is polystable if (i) and (ii) hold from prev thm.

Problem Sheet 8, Exercise 1:

Q: Let $G_m = \operatorname{Spec} k[t,t^{-1}]$ act on $\mathbb{P}^3$ linearly by the representation
$$t \longmapsto \operatorname{diag}(t,t,t^{-1},t^{-1}).$$

- Calculate $R(\mathbb{P}^3)^{G_m}$ for this action.
- Determine X^{ss} and X^{s}.
- Describe $\varphi: X^{ss} \longrightarrow X/\!/G$.

Ans: Action: $t \cdot [x_0 : x_1 : x_2 : x_3] = [tx_0 : tx_1 : t^{-1}x_2 : t^{-1}x_3]$

Let $f \in R(\mathbb{P}^3)^{G_m}$. Then $f = \sum a_{\underline{m}} x_0^{m_0} \cdots x_3^{m_3}$ and $t \cdot f = f$ gives

$$\sum a_{\underline{m}} t^{m_2+m_3-m_0-m_1} x_0^{m_0} x_1^{m_1} x_2^{m_2} x_3^{m_3} = \sum a_{\underline{m}} x_0^{m_0} \cdots x_3^{m_3}$$

$$\Rightarrow \forall\ \underline{m} = (m_0, \dots, m_3),\ \text{either } a_{\underline{m}} = 0 \text{ or } m_0 = m_2 + m_3 - m_1$$

$$\therefore f = \sum a_{\underline{m}} x_0^{m_2+m_3-m_1} x_1^{m_1} x_2^{m_2} x_3^{m_3}$$

$$= \sum a_{\underline{m}} \left(\frac{x_1}{x_0}\right)^{m_1} (x_0x_2)^{m_2} (x_0x_3)^{m_3}$$

Since $\underset{y_1}{\frac{x_1}{x_0}}$, $\underset{y_2}{x_0x_2}$, $\underset{y_3}{x_0x_3}$ are G_m-inv., we see that

$$R(\mathbb{P}^3)^{G_m} = k\left[\frac{x_1}{x_0}, x_0x_2, x_0x_3\right] \cong k[y_1,y_2,y_3]$$

$$\therefore X/\!/G = \operatorname{Proj}\left(R(\mathbb{P}^3)^{G_m}\right) \cong \mathbb{P}^2.$$

- $X^{ss} = \mathbb{P}^3 \setminus N$

$$N = V\left(\frac{x_1}{x_0}, x_0x_2, x_0x_3\right)$$

$$= \left\{ [x_0 : x_1 : x_2 : x_3] \;\middle|\; \frac{x_1}{x_0} = 0,\ x_0x_2 = 0,\ x_0x_3 = 0 \right\}$$

$$= \left\{ x_1 = 0 = x_2 = x_3 \right\}$$

- Therefore, $X^{ss} = \left\{ [x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3 \;\middle|\; x_1 \neq 0 \text{ or } x_2 \neq 0 \text{ or } x_3 \neq 0 \right\}$.

- If $x \in X^{ss}$, then $G_x = \{ t \in k^* \text{ such that } t \cdot [x_0 : x_1 : x_2 : x_3] = [x_0 : x_1 : x_2 : x_3] \}$
" $[x_0 : x_1 : x_2 : x_3]$

$\Rightarrow [tx_0 : tx_1 : t^{-1}x_2 : t^{-1}x_3] = [x_0 : x_1 : x_2 : x_3]$

$\Rightarrow [t^2x_0 : t^2x_1 : x_2 : x_3] = [x_0 : x_1 : x_2 : x_3]$

$\Rightarrow t^2 = \pm 1$

$\Rightarrow G_x$ is finite and hence 0-dim

- Again, Orbit of $x = G \cdot x = \left\{ [tx_0 : tx_1 : t^{-1}x_2 : t^{-1}x_3] \;\middle|\; t \in k^* \right\}$

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Ans: Action: $t\cdot[x_0:x_1:x_2:x_3] = [tx_0:tx_1:t^{-1}x_2:t^{-1}x_3]$

Let $f \in R(\mathbb{P}^3)^{G_m}$. Then $f = \sum a_{\underline{m}} x_0^{m_0} \cdots x_3^{m_3}$ and $t\cdot f = f$ gives

$$\sum a_{\underline{m}} t^{m_2+m_3-m_0-m_1} x_0^{m_0}x_1^{m_1}x_2^{m_2}x_3^{m_3} = \sum a_{\underline{m}} x_0^{m_0}\cdots x_3^{m_3}$$

$$\Rightarrow \forall\ \underline{m}=(m_0,\dots,m_3),\ \text{either } a_{\underline{m}} = 0 \text{ or } m_0 = m_2+m_3-m_1$$

Claim: $R(\mathbb{P}^3)^{G_m} = k[x_0x_2,\ x_0x_3,\ x_1x_2,\ x_1x_3]$

Pf: ($\supseteq$): Is true since $x_0x_2,\ x_0x_3,\ x_1x_2,\ x_1x_3$ are G-inv

($\subseteq$): Suffices to check if $m_0+m_1-m_2-m_3=0$,

then $\underline{x}^{\underline{m}} \in k[x_0x_2,\ x_0x_3,\ x_1x_2,\ x_1x_3]$:

want: $\underline{x}^{\underline{m}} = (x_0x_2)^{c_1}(x_0x_3)^{c_2}(x_1x_2)^{c_3}(x_1x_3)^{c_4}$ w/ $\underline{c} \geq \underline{0}$.

$$\left[\begin{array}{cccc|c} 1 & 1 & & & m_0 \\ & & 1 & 1 & m_1 \\ 1 & & 1 & & m_2 \\ & 1 & & 1 & m_3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & & & m_0 \\ & & 1 & 1 & m_1 \\ -1 & & 1 & & m_2-m_0 \\ & 1 & & 1 & m_3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & & & m_0 \\ & 1 & & 1 & m_3 \\ & & 1 & 1 & m_2-m_0+m_3 \\ & & 1 & 1 & m_1 \end{array}\right]$$

$\therefore$ $c_4 = t$, $c_3 = m_2-m_0+m_3-t$, $c_2 = m_3-t$, $c_1 = m_0-m_3+t$

$$\underline{c} \geq 0 \Leftrightarrow \begin{cases} t \geq 0 \\ t \leq m_2+m_3-m_0 \\ t \leq m_3 \\ t \geq m_3-m_0 \end{cases}$$

Case ①: $m_2-m_0 \geq 0$, can take $t = m_3$

Case ②: $m_2-m_0 < 0$, can take $t = m_1$ (since $m_1 - m_3 = m_2-m_0$) $\square$

Check!

$$N = V(x_0x_2, x_0x_3, x_1x_2, x_1x_3) = V(x_0, x_1) \cup V(x_2, x_3)$$

$X^{ss} = X - N.$

X^s : left as exercise, may use this fact (?)