Throughout today, assume G reductive $X = \operatorname{Spec} R$ affine with $G \curvearrowright X$.

(may specify different notion of reductive) (Recall: always assume finite type throughout reading)

Last time:

Def (4.27) The affine GIT quotient is the map $\varphi: X \to X/\!\!/G := \operatorname{Spec} R^G$ $(= \operatorname{Spec}(\mathcal{O}(X))^G)$

Rmk: Nagata's theorem $\Rightarrow R^G$ is f.g. as a k-algebra.

$\therefore$ $X/\!\!/G$ is finite type over k.

Running example (3.17): $G = G_m$, $X = \mathbb{A}^2$, $G \curvearrowright X$ by

$$t \cdot (x,y) = (tx, t^{-1}y)$$

Then $G \curvearrowright \mathcal{O}(X)$ by

$$t \cdot x = t^{-1}x$$

$$t \cdot y = ty$$

We saw in week 5 that $\mathbb{A}^2 \to \mathbb{A}^1$, $(x,y) \mapsto xy$ is a good quotient,

but the target $\mathbb{A}^1$ and the map $(x,y) \mapsto xy$ came up (somewhat) arbitrarily.

Now: Know there is a natural choice of quotient:

$$X \to X/\!\!/G := \operatorname{Spec} k[x,y]^G = \operatorname{Spec} k[xy]$$

with map given by $k[xy] \hookrightarrow k[x,y]$

$xy \mapsto xy$ (inclusion)

Want to show: Affine GIT quotient is good quotient in general

Before that:

Lemma (4.29) Let G geom. red. group. $G \curvearrowright X$.
Suppose $W_1, W_2 \subseteq X$ are disjoint G-inv closed,
then $\exists f \in \mathcal{O}(X)^G$ which separates these sets, i.e.
$f(W_1) = 0, \quad f(W_2) = 1$.

Proof Step ①: Find $f_1 \in \mathcal{O}(X)$ s.t. $f_1(W_1) = 0, \ f_1(W_2) = 1$
(f_1: not G-inv.)

Construction: $W_1 \cap W_2 = \emptyset$

$\Rightarrow (1) = I(\emptyset) = I(W_1 \cap W_2) \overset{\text{Nullstellensatz}}{=} \sqrt{I(W_1) + I(W_2)} \overset{\sqrt{I} = (1) \Rightarrow 1 \in I}{=} I(W_1) + I(W_2)$

$\therefore$ Can write $1 = f_1 + f_2$ w/ $f_1 \in I(W_1), \ f_2 \in I(W_2)$.
(1: const. function on X)

$\therefore \ f_1(W_1) = 0, \ f_1(W_2) = 1$

Step ② Want function $h: X \to \mathbb{A}^n$ s.t. $h(W_1) = 0$ and $h(W_2) \neq 0$

Construction: Lemma 3.8 $\Rightarrow G \curvearrowright \mathcal{O}(X)$ is rational.
(Affine $G \curvearrowright$ Affine X)

$\therefore \exists$ vec. sp. $V \subseteq \mathcal{O}(X)$ s.t. $\begin{cases} V \text{ is } G\text{-inv} \\ f_1 \in V \\ \dim V < \infty \end{cases}$

$\Rightarrow \forall$ f.d. vec. sp. $W \subseteq \mathcal{O}(X)$, $\dim \langle G \cdot W \rangle < \infty$
(Take $W = \langle f_1 \rangle$)

$\therefore \dim \langle g \cdot f_1 : g \in G \rangle < \infty$
($V :=$ ↓)

Take basis: $V = \langle h_1, \dots, h_n \rangle$ $(n := \dim V)$

$h_i \in \langle g \cdot f_1 \rangle \Rightarrow h_i = \sum_{\ell=1}^{n_i} c_{i,\ell} (g_{i,\ell} \cdot f_1)$ for $c_{i,\ell} \in k, \ g_{i,\ell} \in G$

Let $h: X \to \mathbb{A}^n$ by

$$x \mapsto (h_1(x), \dots, h_n(x))$$

For $x \in X$, have $h_i(x) = \sum_{\ell=1}^{n_i} c_{i,\ell} f_i(g_{i,\ell}^{-1} \cdot x)$

Note: $\begin{cases} g_{i,\ell}^{-1} \cdot W_1 \subseteq W_1 \\ g_{i,\ell}^{-1} \cdot W_2 \subseteq W_2 \end{cases}$

$\Rightarrow h_i(W_1) = 0, \; h_i(W_2) = \sum_{\ell=1}^{n_i} c_{i,\ell}$

$\Rightarrow h(W_1) = \vec{0}, \; h(W_2) = v \neq 0 \quad (\text{w/ } v_i = \sum_{\ell=1}^{n_i} c_{i,\ell})$

$f_1 \in V \Rightarrow f_1 = \sum_{i=1}^{n} d_i h_i$

If $v = 0 \Rightarrow h_i(W_2) = 0 \Rightarrow f_1(W_2) = \sum_{i=1}^{n} d_i h_i(W_2) = 0$ contradiction

Step ③ Find $f \in \mathcal{O}(X)^G$ s.t. $f(W_1) = 0, \; f(W_2) = 1$

★ Using geom. red. in this step. ★

Let $G \curvearrowright \mathbb{A}^n$ as follows:

Write $g \cdot h_i = \sum_{j=1}^{n} a_{ij}(g) h_j \qquad a_{ij} \in \mathcal{O}(G), \; a_{ij}(g) \in k$

This defines a rep $G \to GL(\mathbb{A}^n)$ by

$$g \mapsto (a_{ij}(g))$$

Then $h: X \to \mathbb{A}^n$ is G-equivariant:

$$h(g \cdot x) = \begin{pmatrix} h_1(g \cdot x) \\ \vdots \\ h_n(g \cdot x) \end{pmatrix} \qquad g \cdot (h(x)) \overset{?}{=} \begin{pmatrix} \sum_{j=1}^{n} a_{ij}(g) h_j(x) \end{pmatrix} \overset{?}{=} \begin{pmatrix} (g \cdot h_i)(x) \end{pmatrix}$$

Then $v = h(W_2) \neq 0$ is G-inv.

geom. red. $\Rightarrow \exists$ non-constant homog. $P \in k[x_1, \dots, x_n]^G$ s.t.

$P(v) \neq 0 \quad \Rightarrow P(0) = 0$

$\therefore$ Take $f = cP \circ h$, then it is

- G-inv:

$$cP \circ h(g \cdot x) = cP(g \cdot h(x)) = cP(h(x))$$

- $f(W_1) = 0, \quad f(W_2) = 1$

$h(W_1) = 0 \Rightarrow cP(h(W_1)) = 0$

$h(W_2) = v \Rightarrow P(h(W_2)) = P(v) \Rightarrow cP(h(W_1)) = 1$ $\square$

Thm (4.30) Let G red. $G \curvearrowright X$.

Then the affine GIT quotient $\varphi : X \to X/\!/G$ is a good quotient.

Moreover, $X/\!/G$ is affine (of finite type $/k$)

Proof Nagata's thm $\Rightarrow Y := X/\!/G = \operatorname{Spec} \mathcal{O}(X)^G$ affine of finite type $/k$.

Recall: Good quotient if

(i) φ is G-inv

(ii) φ is surj.

(iii) If $U \subseteq Y$ open, then $\mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$ is an isom. onto $\mathcal{O}_X(\varphi^{-1}(U))^G$

(v') If $W_1, W_2 \subseteq X$ disjoint G-inv. closed, then $\overline{\varphi(W_1)} \cap \overline{\varphi(W_2)} = \emptyset$.

(vi) φ is affine.

(i): $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$ automatically G-inv.

(vi): (Stacks Project 29.11.13) Map between aff. schemes is affine.

(ii): Let $y \in Y(k)$ a closed point. Want to find preimage.

Let $m_y \subseteq \mathcal{O}(Y) = \mathcal{O}(X)^G$ max ideal of y.

Choose generators $m_y = (f_1, \dots, f_m)$

Claim: $\sum_{i=1}^{m} f_i \mathcal{O}(X) \subsetneq \mathcal{O}(X)$

Proof if G lin. red.: (If G geom. red., see [31] Lemma 3.4.2)

Lemma 4.24 $\Rightarrow$ $\sum_{i=1}^{m} f_i \mathcal{O}(X) \cap \mathcal{O}(X)^G = m_y \neq \mathcal{O}(X)^G$

(make use of Reynold) $\therefore \sum_{i=1}^{m} f_i \mathcal{O}(X) \neq \mathcal{O}(X)$

$\therefore \exists\, m_x \subseteq \mathcal{O}(X)$ s.t. $\varphi(x) = y$. $\left(i : \mathcal{O}(X)^G \to \mathcal{O}(X),\ m_y = i^{-1}(m_x) \right)$

$\therefore \varphi$ is surj. onto closed points

Fact: Chevalley's thm: (Hartshorne I Ex 3.19) Let $\varphi : X \to Y$ morphism of schemes, then $\varphi(X)$ is constructible:

Def-proposition (Hartshorne I Ex 3.18(a)) A subset $S \subseteq Y$ is constructible if it can be written as a finite union of locally closed sets

intersection of a open set and a closed set.

$O(X)^G \hookrightarrow O(X)$ injective $\Rightarrow \varphi$ dominant $\Rightarrow \overline{\varphi(X)} = Y$.

(Don't know how to show) $\ldots \Rightarrow \varphi$ surjective.

(iii) Since $U = Y_f$ ($= \{y : f(y) \neq 0\}$) form a basis of open subsets of Y, suffices to look at $\varphi^{-1}(Y_f) \to Y_f$ for $f \in O(X)^G$.

($\varphi^{-1}(Y_f) = X_f$)

$$O_X(\varphi^{-1}(Y_f))^G = (O(X)_f)^G \overset{(1)}{=} (O(X)^G)_f = O(Y)_f$$

$(R_f)^G = (R^G)_f$

$\therefore O_Y(Y_f) \to O_X(\varphi^{-1}(Y_f))$ is isom.

(v'): Lemma 4.29 $\Rightarrow \exists\ f \in O(X)^G$ s.t. $f(W_1) = 0$, $f(W_2) = 1$.

($\uparrow$ regular func. on Y.)

Treat $f : Y \to \mathbb{A}^1_k$

$\therefore f(\varphi(W_1)) = 0$ and $f(\varphi(W_2)) = 1$

$\varphi(W_1) \subseteq f^{-1}(0)$, $\varphi(W_2) \subseteq f^{-1}(1)$

$\Rightarrow \overline{\varphi(W_1)} \subseteq f^{-1}(0)$, $\overline{\varphi(W_2)} \subseteq f^{-1}(1)$ ($\because f^{-1}(0), f^{-1}(1)$ closed)

$\Rightarrow \overline{\varphi(W_1)} \cap \overline{\varphi(W_2)} = \phi$ $\square$

Recall Cor 3.32 Suppose $G \curvearrowright X$, $\varphi : X \to Y$ good quotient:

(a) $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \phi \iff \varphi(x_1) = \varphi(x_2)$

(b) $\forall y \in Y$, $\varphi^{-1}(y)$ contains a unique closed orbit.

In particular, if action is closed (all orbits closed).

then φ is geom. quotient.

Example of GIT quotient:

① $G_m \curvearrowright \mathbb{A}^2$ by

$t \cdot (x,y) = (tx, t^{-1}y)$

$\varphi : \mathbb{A}^2 \longrightarrow \mathbb{A}^1$

$\operatorname{Spec} k[x,y] \longrightarrow \operatorname{Spec} \underbrace{k[xy]}_{\mathcal{O}(X)^G}$

$k[xy] \longrightarrow k[x,y]$

$xy \longmapsto xy$

Note: $\varphi^{-1}(0) =$ $\dagger$ union of 3 orbits:

$\circ \cup \{\} \cup \bullet$.

$\therefore$ Not geom quot.

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② $G_m \curvearrowright \mathbb{A}^n$ by

$t \cdot (x_1, \dots, x_n) = (tx_1, \dots, tx_n)$

$\varphi : \mathbb{A}^n \longrightarrow \operatorname{Spec} \underbrace{k}_{\mathcal{O}(X)^G}$

$k \longrightarrow k[x_1, \dots, x_n]$

$1 \longmapsto 1$

$\varphi^{-1}(0) =$ union of all orbits

$\therefore$ Not geom quot.

(Problem: 0 lies in every orbit closure, wants to get rid of it)

Rmk (4.34)
- Used geom. reductive to show $\mathcal{O}(X)^G$ f.g. (Nagata)
  (Hilbert, Mumford: used lin. red.)
- Used lin. red. to show Lemma 4.24,
  which implies $X \to X/\!/G$ surj.
- Used geom. reductive to show Lemma 4.29
  (some separation statement)